THE TEMPORAL CONCEPTION: STUDENT DIFFICULTIES DEFINING PROBABILISTIC INDEPENDENCE

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This article discusses results from interviews investigating students' understanding of probabilistic independence and mutual exclusivity. Three students compared several sets of events in various sample spaces. Data collected from these interviews gives evidence of a temporal conception wherein students think of independence as reliant on a chronological sequence of events and conditioning. With this approach, students may see some sample spaces (e.g. spinners) as having pair-wise independence for all events. In other sample spaces, (e.g. decks of cards) students see pairs of events as both independent and not independent, this being determined by whether replacement occurs.

Introduction

Probability and Statistics have been increasingly emphasized in elementary and high school education over the past two decades (National Council of Teachers of Mathematics [NCTM], 1989; NCTM, 2000). This emphasis has also extended into post-secondary education. Mutual exclusivity and independence have gained such emphasis, making it important to accurately gauge students' conceptualizations of these two ideas. This article discusses results from interviews aimed to better understand how students think about independence and mutual exclusivity in turn contributing to the theoretical framework of how we approach learning in this area.

Manage and Scariano (2010) found that an alarmingly high percentage of undergraduate students who were enrolled in a course in probability and statistics had fundamental misunderstandings about the relationship between the ideas of independent events and mutually exclusive events. They found this by directly assessing students' understanding of this relationship through a non-scientific, multiple-choice survey of 217 students.

The researchers explored student responses to two questions. In each question, two events, A and B, are assumed to have nonzero probabilities in the same sample space. The first question presented a Venn diagram of the sample space with non-intersecting areas, labeled "*A*" and "*B*," within a rectangular sample space. The question stated that *A* and *B* are mutually exclusive and gave four responses: "*A* and *B* are independent events," "*A* and *B* are not independent events," "*A* and *B* may or may not be independent events," and "I really don't know how to do this problem" (Manage & Scariano, 2010, p. 18). The second question presented the fact that *A* and *B* are independent, have nonzero probabilities, included no diagram, and gave four responses: "*A* and *B* are mutually exclusive events," "*A* and *B* are not mutually exclusive events," "*A* and *B* may or may not be mutually exclusive events," and "I really don't know how to do this problem" (Manage & Scariano, 2010, p. 19).

In the first question, 68.3% of students incorrectly chose the first answer choice, namely "*A* and *B* are independent events." In total, 88% of students incorrectly answered this question. With respect to the second question, 36% of students gave the incorrect first response, "*A* and *B* are mutually exclusive events" whereas 23.3% of students responded correctly. By the results of the first question, students seem to think that these two ideas have a direct relationship, that mutual exclusivity implies independence. This misconception seems less prevalent in the second

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question, since the responses were more evenly distributed than they were in the first (Manage & Scariano, 2010).

D'Amelio (2009) found that most participants could not correctly identify a method for calculating the probability of the union of mutually exclusive events. Most students mistook the product, rather than the sum, of the two events' respective probabilities for the proper calculation. These results suggest students' confusion about the use of such a product when calculating certain probabilities. They also point to a similar misconception to that found in Manage and Scariano (2010), particularly a misunderstanding of the distinction between independent and mutually exclusive events. Confusion between calculating the probability of the intersection of two events rather than their union provides an alternative explanation for this mistake.

Shaughnessey (1992) identifies two equivalent definitions of independent events given that the two events A and B are in the same sample space and have nonzero probabilities:

Definition 1- $P(A \mid B) = P(A)$	**Definition 2-** $P(A \cap B) = P(A) \times P(B)$.

Table 1. Definitions of Independence

Both D'Amelio (2009) and Manage and Scariano (2010) use Definition 2 in their research. Each of these researchers also define two events A and B as "*mutually exclusive* if and only if (A and B) = $A \cap B = \varnothing$" (p. 15). From this definition, if A and B are mutually exclusive then $P(A \cap B) = 0$. So, by the zero product property, it cannot be the case that $P(A \cap B) = P(A) \times P(B)$ and $P(A \cap B) = 0$ when A and B have nonzero probabilities.

Of the research deliberated, the explicit relationship between independence and mutual exclusivity was found in only three articles (Keeler & Steinhorst, 2001; Kelly & Zwiers, 1988; Manage & Scariano, 2010). Manage and Scariano use the reasoning discussed above whereas Kelly and Zwiers address this relationship in the context of student misunderstanding. They provide several examples of how each of these ideas can be explored separately in a classroom. Kelly and Zwiers contend that, "most of the confusion arises because we, as instructors, do not take the time to relate the two concepts." They then blatantly state the relationship between the two ideas- "mutually exclusive events are (almost) never independent." They attribute the "almost" in this last quote to the "pathological cases" when one or both events considered have zero probability (Kelly & Zwiers, 1988). Keeler et al., however, acknowledge the relationship as a common misunderstanding among students.

In considering pedagogical implications for this research, we find that students have many difficulties with both conditional probability and independence (Shaughnessey, 1992). This research goes on to say that students' "misconceptions of conditional probability may be closely related to students' understanding of independent events and of randomness in general" (Shaughnessey, 1992, p. 475). He points out that many researchers "advocate introducing the concept of independence via the conditional probability definition (Definition 1), as they believe this is more intuitive for students" (1992, p. 475). This intuition comes in the context of without-replacement problems. If the sample space remains unchanged, then the first experiment bears no affect on the second experiment.

Other pedagogical research in this area discusses students' misconceptions related to independence almost exclusively with respect to conditional probability (Tarr & Lannin, 2005). Tarr and Lannin (2005) justify their concentration on these types of misconceptions, citing Shaughnessey (1992), and focus on replacement and non-replacement situations because of the

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prevalence of these types of problems in the typical curriculum. Tarr and Lannin state that "within this context [that of *with-replacement* situations], an 'understanding of independence' is demonstrated by students' ability to recognize and correctly explain when the occurrence of one event does not influence the probability of another event" (2005, p. 216). There is also an emphasis that students understand the change of an event's probability in non-replacement conditional probability problems is due to the change of the sample space.

While Shaughnessey (1992), Tarr and Lannin (2005), and Keeler and Steinhorst (2001) suggest conditional probability as a context for independence D'Amelio (2009), Kelly and Zwiers (1988), and Manage and Scariano (2010) each explore student misconceptions outside of the conditional probability setting. This could provide some reasoning into why D'Amelio (2009) and Manage and Scariano (2010) had such disturbingly low numbers of correct responses. Supposing that the students' previous curricula addressed independence in the context of conditional probability, the students may not have been able to correctly reason about the relationship between independence and mutual exclusivity without such a context. Perhaps this is what Kelly and Zwiers are arguing when they say that, "we, as instructors, do not take the time to relate the two concepts," referring to mutual exclusivity and independence (1988, p. 100).

Since independence can be defined outside of a conditional probability setting, it is not limited to an order of experiments. This can be seen in a standard deck of cards. The event of drawing a spade is independent of the event of drawing an ace. We can see this since P(spade) = ¼, P(ace) = 1/13, and P(ace ∩ spade) = 1/52 = 1/4 × 1/13. This example demonstrates that independence can be accessibly thought of outside the context of conditional probability. This can prove useful since mutual exclusivity is also defined without respect to time. For instance, in the above case, it can quickly be demonstrated that P(heart∩spade) = 0 since this intersection is empty; so these two events are mutually exclusive.

Altogether, we see that different researchers emphasize two different contexts for the independence of two events. Some focus on conditional probability for the definition of independence (Definition 1), while others focus on the product definition (Definition 2). Shaughnessey (1992) provides an explanation of why educators and some researchers focus on conditional probability when dealing with independence in that it is more intuitive for students to explore independence in the context of conditional probability. Regardless, students' misconceptions about the relationship between independence and mutual exclusivity prevail.

Methods

Interviews were conducted with three undergraduate students who were enrolled in a Junior-level Proofs course (Alex, Betty, and Caroline). Betty had recently taken an undergraduate course in probability and statistics, whereas Alex and Caroline had not. In these interviews, the students were asked to give definitions of independent events and mutually exclusive events as well as provide examples of each. The students were then given various sample spaces and events within those sample spaces and asked to determine whether pairs of events were independent. The purpose of these interviews as part of broader research was to gauge the students' understanding of these concepts and the relationship between the two. Two initial sample spaces were discussed and then others were explored as students and the interviewer responded to situations throughout the interviews.

The first sample space consisted of a standard deck of cards and a fair six-sided die. The events discussed were the simultaneous drawing of a card and rolling of the die. For example, event A was "drawing a spade and rolling any number on the die." Event B was "drawing any

card and rolling a three on the die." This sample space is made up of two smaller sample spaces (let's call them "subsample spaces") that are often considered individually. Each of these subsample spaces is independent of the other, as the card drawn would have no effect on the die and vice versa. The second sample space was similar to the first in that it consisted of two subsample spaces. In this sample space, an event consisted of tossing a fair coin (heads and tails) and spinning a spinner with five colors (blue, red, green, orange, and yellow) with given probabilities (.3, .2, .2, .2, and .1, respectively). So, for example, event A was tossing a head and spinning blue.

These sample spaces were chosen to observe how readily students identified the subsample spaces as independent and how they thought of the events in these sample spaces with respect to independence and mutual exclusivity. The complexity of each sample space also provided somewhat familiar situations, the combination of which prevented the participants from simply remembering probabilities from previous experience. After several pairs of events in each sample space were discussed, the interviewer and participant each introduced different sample spaces in discussion. For instance, Betty discussed a deck of cards without a die and the interviewer brought up the sample space of a die without the deck of cards when interviewing Caroline. This allowed the interviewer and participant to discuss caveats and nuances of the concepts of independence and mutual exclusivity in their own terms.

Results

Alex defined independence as, "[when] the outcome of one event does not affect the outcome of a subsequent event." This definition implies an emphasis on a sequence of events, where one of the events being considered must occur prior to the other. With regard to mutual exclusivity, however, Alex was less certain of a formal definition- changing his phrasing twice throughout the interview and eventually declaring, "Performing an event or series of events causes a subsequent event to have zero probability of happening." Again, Alex implies that this relationship is defined over a period of time. When prompted for an example of independent events, Alex gave the example of a die. He stated that rolling a six on the first roll of a die does not affect rolling a six on the second roll of a die. This example is consistent with his definition, implying that the two events in consideration take place at separate times.

Betty stated that, "Two events are independent if the probability of A occurring does not affect the probability of B occurring." Betty also described the independence of events A and B using the equation $P(A) = P(A|B)$. In comparison to Alex, this definition of independence does not necessarily imply that one event must occur before the other. But, when prompted for an example of independent events, Betty described the act of picking a card from a deck of fifty-two cards, and putting it back so that the probability of picking a second card is not affected. Similarly, when asked for an example of events *not* being independent, Betty provided the case of picking a card and not replacing it. These examples are consistent with the notion of independence in the context of a "with replacement" and "without replacement" conditioning event. In contrast, Betty defined mutually exclusive events with the statement, "you can't have both at the same time." This definition explicitly states that the events can be compared instantaneously. Here, Betty gave the example that the queen of hearts and jack of diamonds are mutually exclusive, since they cannot both occur when one card is drawn.

Caroline's definition for independence was similar to the other two participants, stating, "Two events are independent if they do not affect each other." Caroline's example of independent events was different from both Alex's and Betty's in that Caroline described "everyday events" rather than "artificial" events (such as dice or cards) that are typically

investigated in the classroom setting. Caroline described how "the probability of someone wearing a red shirt is independent of their age." Similarly, when prompted for an example of events that are not independent, Caroline provided the example of someone who is forgetful is less likely to win a student lottery for a football ticket, since they are less likely to enter the lottery. This example implies a directly causal relationship, where the lower probability of the first event (entering the lottery) decreases the probability of a later event (being selected in said lottery).

It should be noted that all three participants showed initial difficulty in differentiating between the concepts of independence and mutual exclusivity, although each did eventually distinguish between the concepts. Alex, for instance, initially stated that the two concepts are "more or less the same thing" and "pretty much synonymous." Betty's first definition for mutually exclusive events, which she quickly changed, was $P(A \cap B) = P(A)*P(B)$, the mathematical definition of independence. Caroline had greater difficulty defining mutual exclusivity, stating, "It sounds like they would be independent of one another." From this, we see at least initial difficulty distinguishing between these two concepts among all the participants, consistent with the literature (Manage & Scariano, 2010).

Independence in the Sample Spaces

In the first sample space, the participants explored two pairs of events (Table 2). All three participants reacted in similar ways to the question, "Are these events independent?" In each case, all three students established various cases, beginning with some form of the question, "Do you put the card back?" In the first case, the participant described successfully performing one event, putting the card back, reshuffling (or resetting) the deck, and then successfully performing the other event in question. In each interview, the participant claimed that the events would be independent in this case. In the second case, the sequence of events was identical, except that the card was not replaced and the deck was not reshuffled.

Answers were slightly more varied in this second case. For instance, Alex thought that neither pair of events was independent, since you "change the context," but only considered event A before event B and event C before event D. Betty considered whether A came before B or vice versa and similarly for events C and D. In the first pair, Betty responded that A before B gave independence, but B before A did not. With the second pair, each event would change the probability of the other, causing C to always be not independent of D. Caroline generalized all pairs of events under this case, explaining that without replacement you affect the probability of the second event, so that no two events will be independent without replacement.

		Card	Die
Pair 1	Event A	Spade	"Any Number"
	Event B	"Any Card"	3
Pair 2	Event C	Spade	3
	Event D	Heart	4

Table 2. Events in Sample Space 1

Event B produced the most diverse explanations. Alex seemed to think of event B as impossible after drawing any card without replacing it. He attributed this to the fact that the first card would be a member of "any card," so that drawing any card would not be possible, since you will have removed one of them. Betty thought of Event B as always successful and independent. Caroline thought of event B as always successful as long as a card was remaining,

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but independent only when the first card was replaced. It should be noted also, that explanations of whether events on the die were independent were almost nonexistent. All three participants focused on whether the card was replaced. Alex even said, "With the die, it really doesn't matter what you do… If you're just rolling dice over and over, those are always independent."

The second sample space provided much more homogeneous responses among the participants. Most of the events discussed were single outcomes from each subsample space (i.e. one color from the spinner and one side of the coin). A few events discussed were similar to events A and B from the first sample space, where a compound event from one of the subsample spaces was considered. When asked to compare any two events in the sample space, all three participants thought of every pair of events as independent from each other. This reflects the lack of emphasis on the die in the first sample space. Successfully completing an event on both a coin and a spinner does not physically remove an object from the sample space. This fact has strong implications about the students' understanding of the relationship between independence and mutual exclusivity.

Mutual Exclusivity in the Sample Spaces

The participants seemed to struggle much more with determining two events' mutual exclusivity than with their independence. Alex changed his definition of mutually exclusive events twice, each time after encountering examples that challenged his definition. Initially, Alex's definition seemed consistent with the mathematical definition. He explained a scenario of it raining or not raining, saying, "It couldn't be raining and not raining at the same time." This phrasing alludes to an empty intersection. Throughout the interview, Alex changed his definition to more closely align with how he discussed independence, so that one event would cause a subsequent event to be impossible. Interestingly, as a corollary to his last definition of mutual exclusivity, Alex pointed out that two mutually exclusive events couldn't be independent. He made this connection before the interviewer discussed any relationship between the two terms.

Because he changed his definition, Alex's responses in the beginning of the interview convey a different conceptualization of mutual exclusivity than do his later responses. When comparing events C and D he determined that these two events are not mutually exclusive since drawing a spade does not cause drawing a heart to have zero probability. In his exploration of the second sample space, Alex concludes that no two events are mutually exclusive. This relies on his responses regarding independence, since no event in the second sample space changed the probability of a subsequent event, no event could cause a subsequent event to have zero probability.

Betty's responses to questions of mutual exclusivity were consistent with the correct mathematical definition in both sample spaces. This is because her definition of the concept was a typical translation into "everyday" language, "You can't have them both at the same time." This is not too surprising since Betty was the only participant to have had recent instruction in probability, even though Betty showed early confusion about the distinction between independence and mutual exclusivity.

Caroline also had difficulty establishing a definition for mutual exclusivity. Her final definition, "Two events that do not include parts of each other," is very similar to the mathematical definition. Most of Caroline's responses in each sample space were consistent with correct responses, except two responses in the second sample space. Event A in this sample space was "spinning blue on the spinner and tossing heads on the coin" and event E was "spinning blue or yellow and tossing heads." Caroline concluded that event A is not mutually exclusive of event E, since it was a subset of event E. But Caroline claimed that event E is

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mutually exclusive of event A, since you could successfully perform event E and not event A by spinning a yellow and tossing heads.

The Temporal Conception

We see from these results that students think about independence of two events by assuming the occurrence of one event and then considering whether this changes the probability of a second event that occurs chronologically after the first. The tendency for students to think of independence in this way causes a temporal conception. With all three participants, questions of events' independence were answered with students considering sequences of events. With Alex, though not in the case of Betty or Caroline, this temporal conception was also evident in his definition of mutually exclusive events.

It should be noted that, had the interviewer asked the participants to determine if the first event in a sequence of trials had changed the probability of a second event, the responses that the participants gave would have been generally correct. The misconception is that examples of chronological dependence and independence generalize to the standard definition of probabilistic independence. For instance, in the second sample space, Alex viewed all events as independent and not mutually exclusive. Meanwhile, Betty and Caroline viewed all events in the second sample space as independent, with some pairs being mutually exclusive and some not. In reality, the majority of the events discussed were mutually exclusive of each other (e.g. blue/heads, red/tails, orange/heads, etc.) and therefore not independent since most had nonzero probabilities. Furthermore, in the first sample space all participants found at least some pairs of events to be both independent (with replacement) and not independent (without replacement). This allows students to think of independence as a consequence of time and therefore allows the misconception that the same two events can be both independent and not independent.

Discussion

Understanding this temporal conception could provide insight into how students confuse the concepts of independence and mutual exclusivity. For instance, under this conception, a student could conclude that mutually exclusive events can be both independent and not independent, as was seen in the first sample space with events C and D. One may suggest that the phrase "mutually exclusive" and the word "independent" are somewhat synonymous in their everyday context, as hinted in the literature (Manage & Scariano, 2010). This was evident in the initial responses by each participant. This explanation, however, only helps to explain the responses of those students who mistakenly thought that one term implied the other. The temporal conception could potentially help explain how students thought events could be "mutually exclusive and maybe or maybe not independent" or "independent and maybe or maybe not mutually exclusive."

Kelly and Zwiers (1988) touched on the notion that students had temporal conceptions about independence. The authors discuss statement, "events A and B are independent if knowledge about whether A has occurred provides us with no knowledge about whether has occurred," (1988, p. 98) noting that, "In a very subtle way an element of time is hinted at in such a statement and it often confuses students." (1988, p. 98) They then emphasize the importance that students understand that independent events are independent regardless of time. While I agree with the importance of such a concept, it is not entirely obvious that understanding this will help students avoid the temporal conception. It is equally important that students understand that knowledge of whether events are independent is calculable without any element of time, assuming that one knows the theoretical probability of each event and of their intersection.

It can be argued that this subtle element of time is also present when discussing conditional probability. This is important since much of the research suggests that independence be taught in the context of conditional probability (Shaughnessey, 1992; Tarr & Lannin, 2005; Keeler & Steinhorst, 2001). For instance, the phrase “given that the other event has occurred” implies that perhaps the first event’s occurrence was chronologically prior to the second. Further research could investigate what relationships exist between these concepts with respect to this element of time. It is also important to understand how we as educators can overcome this temporal conception. Analysis of current school mathematics curricula could allow insight into how students build notions of independence. For instance, repeatedly seeing without replacement scenarios of conditional probability as a context for how two events can be “not independent” could enable students to develop the temporal conception with independence.

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